Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Advances in Mathematics 200 (2006) 102–124

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Path model for a level-zero extremal weight module over a quantum affine algebra II

Satoshi Naito, Daisuke Sagaki*

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

Received 8 March 2004; accepted 25 August 2004
Available online 30 December 2004

Abstract

Let ϖ_i be a level-zero fundamental weight for an affine Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$, and let $\mathbb{B}(\varpi_i)$ be the crystal of all Lakshmibai–Seshadri paths of shape ϖ_i. First, we prove that the crystal graph of $\mathbb{B}(\varpi_i)$ is connected. By combining this fact with the main result of our previous work, we see that $\mathbb{B}(\varpi_i)$ is, as a crystal, isomorphic to the crystal base $\mathscr{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ over a quantum affine algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$ of extremal weight ϖ_i. Next, we obtain an explicit description of the decomposition of the crystal $\mathbb{B}(m\varpi_i)$ of all Lakshmibai–Seshadri paths of shape $m\varpi_i$ into connected components. Furthermore, we prove that $\mathbb{B}(m\varpi_i)$ is, as a crystal, isomorphic to the crystal base $\mathscr{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$.

MSC: 17B37; 17B67; 81R50; 81R10

Keywords: Crystal bases; Path models; Extremal weight modules; Quantum affine algebras

0. Introduction

This paper is a continuation of our previous work [NS]. Let us recall the main result of [NS] briefly. Let $\mathfrak{g}$ be an affine Lie algebra over $\mathbb{Q}$ with Cartan subalgebra $\mathfrak{h}$.

* Corresponding author.
E-mail addresses: naito@math.tsukuba.ac.jp (S. Naito), sagaki@math.tsukuba.ac.jp (D. Sagaki).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2004.08.016

Denote by $\Pi := \{\alpha_j\}_{j \in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$ the set of simple roots, and by $\Pi^\vee := \{h_j\}_{j \in I} \subset \mathfrak{h}$ the set of simple coroots, where $I = \{0, 1, \ldots, L\}$ is an index set for the simple roots (we use the numbering of the simple roots as in [Kac], with "L" replacing "ℓ"). Let $c = \sum_{j \in I} a_j^\vee h_j \in \mathfrak{h}$ be the canonical central element of $\mathfrak{g}$. Take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ that contains all the simple roots α_j, $j \in I$, and fundamental weights Λ_j, $j \in I$, for the affine Lie algebra $\mathfrak{g}$. For each $i \in I_0 := I \setminus \{0\}$, we define a level-zero fundamental weight $\varpi_i \in P$ by: $\varpi_i = \Lambda_i - a_i^\vee \Lambda_0$. Note that $\varpi_i(c) = 0$; an integral weight $\lambda \in P$ is said to be level-zero if $\lambda(c) = 0$. Let $\mathbb{P}$ be the set of all paths (i.e., piecewise linear, continuous maps) $\pi : [0, 1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ such that $\pi(0) = 0$ and $\pi(1) \in P$, which is endowed with a crystal structure by (Littelmann's) root operators e_j and f_j, $j \in I$. We define a path $\pi_{\varpi_i} : [0, 1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ by: $\pi_{\varpi_i}(t) = t\varpi_i$, $t \in [0, 1]$, and denote by $\mathbb{B}_0(\varpi_i)$ the connected component of $\mathbb{P}$ containing π_{ϖ_i}, i.e., the set of all paths that are obtained by applying the root operators successively to π_{ϖ_i}. In [NS], we proved that the connected component $\mathbb{B}_0(\varpi_i)$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight ϖ_i, where $U_q(\mathfrak{g})$ is a quantum affine algebra over $\mathbb{Q}(q)$ (see [NS, Theorem 5.1]). Here, an extremal weight module is an integrable $U_q(\mathfrak{g})$-module, which was introduced by Kashiwara [Kas1, §8] as a natural generalization of an integrable highest (or lowest) weight module (see also [Kas3]).

Now, we want a more explicit description of the connected component $\mathbb{B}_0(\varpi_i)$ of $\mathbb{P}$ containing the path π_{ϖ_i}. For an integral weight $\lambda \in P$, we denote by $\mathbb{B}(\lambda)$ the set of all Lakshmibai–Seshadri paths (LS paths for short) of shape λ, which is by definition a set of paths parametrized by pairs of a sequence of elements in $W\lambda$, where W is the Weyl group of $\mathfrak{g}$, and a sequence of rational numbers satisfying a certain condition (see §1.4 below). We know from [L2] that the set $\mathbb{B}(\lambda)$ together with the root operators is a subcrystal of $\mathbb{P}$ containing the path π_λ defined by: $\pi_\lambda(t) = t\lambda$, $t \in [0, 1]$. Hence the connected component $\mathbb{B}_0(\varpi_i)$ is contained in the crystal $\mathbb{B}(\varpi_i)$ of all LS paths of shape ϖ_i. Then, quite a natural question is whether or not the subset $\mathbb{B}_0(\varpi_i)$ is equal to the whole of $\mathbb{B}(\varpi_i)$. In this paper, we first give an affirmative answer to this question. In fact, we prove:

Theorem 1. *The crystal graph of the crystal $\mathbb{B}(\varpi_i)$ of all LS paths of shape ϖ_i is connected. In particular, the set $\mathbb{B}_0(\varpi_i)$ in [NS, Theorem 5.1], which is the connected component of $\mathbb{P}$ containing the straight line $\pi_{\varpi_i}(t) := t\varpi_i$, $t \in [0, 1]$, is equal to $\mathbb{B}(\varpi_i)$.*

By combining Theorem 1 with the main result of our previous work [NS, Theorem 5.1], we obtain:

Corollary. *The crystal $\mathbb{B}(\varpi_i)$ of all LS paths of shape ϖ_i is, as a crystal, isomorphic to the crystal base $\mathcal{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight ϖ_i.*

Next, we study the crystal structure of the crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ for $m \in \mathbb{Z}_{\geqslant 1}$, and obtain an explicit description of the decomposition of the crystal $\mathbb{B}(m\varpi_i)$ into connected components. Furthermore, as an extension of [NS, Theorem 5.1], we prove that the crystal $\mathbb{B}(m\varpi_i)$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$.

Let us explain this result of ours more precisely. Let $\mathrm{Par}_{<m}$ be the set of partitions of length (i.e., number of parts) strictly less than m. For each $\sigma \in \mathrm{Par}_{<m}$ of the form $(k_1, k_2, \dots, k_{m-1})$ with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$, we denote by $|\sigma|$ the weight of σ, i.e., $|\sigma| := k_1 + k_2 + \cdots + k_{m-1}$. We can define a crystal structure on $\mathrm{Par}_{<m}$ as follows:

$$\begin{cases} e_j\sigma = f_j\sigma = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \varepsilon_j(\sigma) = \varphi_j(\sigma) = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \mathrm{wt}(\sigma) = -|\sigma| d_i \delta & \text{for } \sigma \in \mathrm{Par}_{<m}. \end{cases}$$

Here, the $0\,(= e_j\sigma = f_j\sigma)$ is an extra element, which is not contained in $\mathrm{Par}_{<m}$, and d_i is a positive integer determined by: $\{d \in \mathbb{Z} \mid \varpi_i + d\delta \in W\varpi_i\} = \mathbb{Z}d_i$, where δ is the null root of the affine Lie algebra $\mathfrak{g}$.

Theorem 2. *The crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the tensor product $\mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i)$ of the crystal $\mathrm{Par}_{<m}$ above and the crystal $\mathbb{B}_0(m\varpi_i)$, where $\mathbb{B}_0(m\varpi_i)$ is the connected component of $\mathbb{B}(m\varpi_i)$ containing the straight line $\pi_{m\varpi_i}(t) := t(m\varpi_i)$, $t \in [0, 1]$.*

In [Kas3], Kashiwara studied level-zero extremal weight modules over the quantum affine algebra $U_q(\mathfrak{g})$ and their crystal bases, and also presented a conjecture on the crystal structure of the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$ (see [Kas3, Conjecture 13.2]). This conjecture was first proved independently by Beck [B] and Nakajima [N] for the case where $\mathfrak{g}$ is a symmetric, untwisted affine Lie algebra, and then proved for the general case by their joint work [BN]. Combining [BN, Theorem 4.16 (i)] due to Beck and Nakajima with the theorem above and [NS, Corollary 5.3], we obtain the following:

Corollary. *The crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$.*

This paper is organized as follows. In §1, we fix our notation, and recall some basic facts about affine Lie algebras and path models. In §2, we prove the connectedness of $\mathbb{B}(\varpi_i)$ (i.e., Theorem 1) and its corollary above. In §3, we study the crystal structure of $\mathbb{B}(m\varpi_i)$, and then prove Theorem 2 and its corollary above.

1. Preliminaries

1.1. Affine Lie algebras

Let $\mathfrak{g}$ be an affine Lie algebra over the field $\mathbb{Q}$ of rational numbers, and $\mathfrak{h}$ its Cartan subalgebra. Denote by $\Pi := \{\alpha_j\}_{j\in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$ the set of simple roots, and by $\Pi^\vee := \{h_j\}_{j\in I} \subset \mathfrak{h}$ the set of simple coroots, where $I = \{0, 1, \ldots, L\}$ is an index set for the simple roots. Throughout this paper, we use the numbering of the simple roots as in [Kac, §4.8 and §6] (with L replacing ℓ). Let

$$\delta = \sum_{j\in I} a_j\alpha_j \in \mathfrak{h}^* \qquad \text{and} \qquad c = \sum_{j\in I} a_j^\vee h_j \in \mathfrak{h} \tag{1.1.1}$$

be the null root and the canonical central element of $\mathfrak{g}$, respectively. Denote by $W = \langle r_j \mid j \in I\rangle$ the Weyl group of $\mathfrak{g}$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ is the simple reflection in α_j for $j \in I$.

Let Λ_j, $j \in I$, be the fundamental weights for the affine Lie algebra $\mathfrak{g}$, i.e., $\Lambda_j(h_{j'}) = \delta_{j,j'}$ for $j, j' \in I$. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ that contains all the simple roots α_j, $j \in I$, and fundamental weights Λ_j, $j \in I$. For each $i \in I_0 := I \setminus \{0\}$, we define a level-zero fundamental weight $\varpi_i \in P$ by:

$$\varpi_i = \Lambda_i - a_i^\vee \Lambda_0. \tag{1.1.2}$$

Note that $\varpi_i(c) = 0$; an integral weight $\lambda \in P$ is said to be level-zero if $\lambda(c) = 0$.

1.2. Path models

For $a, b \in \mathbb{Q}$ with $a < b$, we set $[a, b]_{\mathbb{Q}} := \{t \in \mathbb{Q} \mid a \leqslant t \leqslant b\}$, and write just $[a, b]$ for $[a, b]_{\mathbb{Q}}$ to simplify the notation. A path is, by definition, a piecewise linear, continuous map $\pi : [0, 1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ such that $\pi(0) = 0$ and $\pi(1) \in P$. Let $\mathbb{P}$ be the set of all paths.

Remark 1.2.1. In [L1,L2,NS], two paths π and π' are identified if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0, 1] \to [0, 1]$ (reparametrizations) such that $\pi \circ \psi = \pi' \circ \psi'$. However, this identification is not essential for the theory of path models. In fact, we can check that many of the results in [L2,NS], which will be used in this paper, still hold without this identification (adopt (3.2.1) below as the definition of a concatenation of paths).

For a path $\pi \in \mathbb{P}$ and $j \in I$, we define

$$\begin{aligned} H_j^\pi(t) &:= (\pi(t))(h_j) \quad \text{for } t \in [0, 1], \\ m_j^\pi &:= \min\{H_j^\pi(t) \mid t \in [0, 1]\}. \end{aligned} \tag{1.2.1}$$

Let $\mathbb{P}_{\mathrm{int}}$ be the subset of $\mathbb{P}$ consisting of all paths satisfying the following condition:

(INT) For every $j \in I$, all local minimums of $H_j^\pi(t)$ are integers (cf. [L2, Lemma 4.5 (d)]).

Remark 1.2.2. (1) It is obvious that for every $\pi \in \mathbb{P}_{\mathrm{int}}$ and $j \in I$, m_j^π is a nonpositive integer, since $\pi(0) = 0$. Furthermore, it readily follows that $H_j^\pi(1) - m_j^\pi$ is a nonnegative integer, since $\pi(1) \in P$ is an integral weight and m_j^π is the minimal value of $H_j^\pi(t)$.

(2) The condition (INT) is satisfied by all paths used in this paper, such as LS paths and concatenations of them (see §1.4 and §3.2 below).

In [L2, §1], Littelmann introduced root operators $e_j : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ and $f_j : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ for $j \in I$, where θ is an extra element, which corresponds to the 0 in the theory of crystals (cf. [Kas2, §7.2]; by convention, $e_j\theta = f_j\theta = \theta$). Then he proved in [L2, §2] that the set $\mathbb{P}$ of all paths together with the root operators e_j and f_j, $j \in I$, has a natural crystal structure. Here we give the description of $e_j\pi \in \mathbb{P}$ and $f_j\pi \in \mathbb{P}$ for $\pi \in \mathbb{P}_{\mathrm{int}}$ (see also [L1, §1]); as mentioned in Remark 1.2.2 (2), all paths used in this paper are elements of $\mathbb{P}_{\mathrm{int}}$. Let $\pi \in \mathbb{P}_{\mathrm{int}}$ and $j \in I$. Then, $e_j\pi \in \mathbb{P}$ is given as follows (see Remark 1.2.2 (1)): If $m_j^\pi = 0$, then $e_j\pi = \theta$. If $m_j^\pi \leqslant -1$, then

$$(e_j\pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leqslant t \leqslant t_1, \\ \pi(t) + \alpha_j & \text{if } t_1 \leqslant t \leqslant 1, \end{cases} \tag{1.2.2}$$

where we set

$$\begin{aligned} t_1 &:= \min\{t \in [0,1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_0 &:= \max\{t' \in [0,t_1] \mid H_j^\pi(t) \geqslant m_j^\pi + 1 \text{ for all } t \in [0,t']\}. \end{aligned} \tag{1.2.3}$$

Similarly, $f_j\pi \in \mathbb{P}$ is given as follows (see Remark 1.2.2 (1)): If $H_j^\pi(1) - m_j^\pi = 0$, then $f_j\pi = \theta$. If $H_j^\pi(1) - m_j^\pi \geqslant 1$, then

$$(f_j\pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \leqslant t \leqslant t_1, \\ \pi(t) - \alpha_j & \text{if } t_1 \leqslant t \leqslant 1, \end{cases} \tag{1.2.4}$$

where we set

$$\begin{aligned} t_0 &:= \max\{t \in [0,1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_1 &:= \min\{t' \in [t_0,1] \mid H_j^\pi(t) \geqslant m_j^\pi + 1 \text{ for all } t \in [t',1]\}. \end{aligned} \tag{1.2.5}$$

Theorem 1.2.3 (*Littelmann [L2, §2]*). *Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{int}}$ such that the set $\mathbb{B} \cup \{\theta\}$ is stable under the action of the root operators e_j and f_j for all $j \in I$. We define*

$$\begin{cases} \mathrm{wt}(\pi) := \pi(1) & \text{for } \pi \in \mathbb{B}, \\ \varepsilon_j(\pi) := \max\{n \geqslant 0 \mid e_j^n \pi \neq \theta\} & \text{for } \pi \in \mathbb{B} \text{ and } j \in I, \\ \varphi_j(\pi) := \max\{n \geqslant 0 \mid f_j^n \pi \neq \theta\} & \text{for } \pi \in \mathbb{B} \text{ and } j \in I. \end{cases} \tag{1.2.6}$$

Then, the set $\mathbb{B}$ together with the root operators and the maps above is a crystal in the sense of [Kas2, §7.2]. *Moreover, we have $\varepsilon_j(\pi) = -m_j^\pi$, and $\varphi_j(\pi) = H_j^\pi(1) - m_j^\pi$ for $\pi \in \mathbb{B}$ and $j \in I$.*

1.3. Weyl group action

Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{int}}$ such that the set $\mathbb{B} \cup \{\theta\}$ is stable under the action of the root operators e_j and f_j for all $j \in I$. For each $j \in I$, we define $S_j : \mathbb{B} \to \mathbb{B}$ by:

$$S_j \pi = \begin{cases} f_j^n \pi & \text{if } n := H_j^\pi(1) \geqslant 0 \\ e_j^{-n} \pi & \text{if } n := H_j^\pi(1) < 0 \end{cases} \quad \text{for } \pi \in \mathbb{B}.$$

Theorem 1.3.1 (*Littelmann [L2, Theorem 8.1]*). *Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{int}}$ such that the set $\mathbb{B} \cup \{\theta\}$ is stable under the action of all the root operators. Then, there exists a unique action $S : W \to \mathrm{Bij}(\mathbb{B})$, $w \mapsto S_w$, of the Weyl group W on the set $\mathbb{B}$ such that $S_{r_j} = S_j$ for all $j \in I$, where $\mathrm{Bij}(\mathbb{B})$ is the group of all bijections from the set $\mathbb{B}$ to itself. Moreover, we have $(S_w \pi)(1) = w(\pi(1))$ for $w \in W$ and $\pi \in \mathbb{B}$.*

1.4. Lakshmibai–Seshadri paths

In this subsection, we recall the definition of Lakshmibai–Seshadri paths from [L2, §4]. For a positive real root β, we denote by r_β the reflection of $\mathfrak{h}^*$ in β, and by $\beta^\vee$ the dual real root of β.

Definition 1.4.1. Let $\lambda \in P$ be an integral weight. For $\mu, \nu \in W\lambda$, we write $\mu \geqslant \nu$ if there exist a sequence $\mu = \xi_0, \xi_1, \ldots, \xi_r = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \ldots, \beta_r$ of positive real roots such that $\xi_k = r_{\beta_k}(\xi_{k-1})$ and $\xi_{k-1}(\beta_k^\vee) < 0$ for $k = 1, 2, \ldots, r$. Then the sequence $\xi_0, \xi_1, \ldots, \xi_r$ above is called a chain for (μ, ν). If $\mu \geqslant \nu$, then we define $\mathrm{dist}(\mu, \nu)$ to be the maximal length r of all possible chains for (μ, ν).

For $\mu, \nu \in W\lambda$, we write $\mu > \nu$ if $\mu \geqslant \nu$ and $\mu \neq \nu$.

Remark 1.4.2. Let $\lambda \in P$, and $\mu, \nu \in W\lambda$ such that $\mu > \nu$ and $\mathrm{dist}(\mu, \nu) = 1$. Let β be a positive real root corresponding to the chain $\mu = \xi_0, \xi_1 = \nu$ for (μ, ν). Then, by Lemma 2.1.4 below, $r_\gamma(\mu) = \nu$ for some positive real root γ (if and) only if $\beta = \gamma$.

We see from the remark above that the following definition of a-chains is equivalent to that in [L2, §4].

Definition 1.4.3. Let $\lambda \in P$, $\mu, \nu \in W\lambda$ with $\mu > \nu$, and $0 < a < 1$ a rational number. An a-chain for (μ, ν) is, by definition, a sequence $\mu = \xi_0 > \xi_1 > \cdots > \xi_r = \nu$ of elements in $W\lambda$ such that $\mathrm{dist}(\xi_{k-1}, \xi_k) = 1$ and $a\xi_{k-1}(\beta_k^\vee) \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \ldots, r$, where β_k is the positive real root corresponding to the chain for (ξ_{k-1}, ξ_k).

Definition 1.4.4. Let $\lambda \in P$, and let $(\underline{\nu}\,;\underline{a})$ be a pair of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\lambda$ and a sequence $\underline{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers. The pair $(\underline{\nu}\,;\underline{a})$ is called a Lakshmibai–Seshadri path (LS path for short) of shape λ, if for every $k = 1, 2, \ldots, s-1$, there exists an a_k-chain for (ν_k, ν_{k+1}). We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ.

Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s\,; a_0, a_1, \ldots, a_s)$ be a pair of a sequence $\nu_1, \nu_2, \ldots, \nu_s \in P$ of integral weights and a sequence $0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers. We associate to the pair π the following path:

$$\pi(t) = \sum_{l=1}^{k-1} (a_l - a_{l-1})\nu_l + (t - a_{k-1})\nu_k \quad \text{for} \quad a_{k-1} \leqslant t \leqslant a_k, \ 1 \leqslant k \leqslant s. \tag{1.4.1}$$

In this way, we can regard the set $\mathbb{B}(\lambda)$ of all LS paths of shape $\lambda \in P$ as a subset of $\mathbb{P}$: $\mathbb{B}(\lambda) \subset \mathbb{P}$. We know the following theorem from [L2, Lemma 4.5 (d) and Corollary 2].

Theorem 1.4.5. *The set $\mathbb{B}(\lambda)$ is a subset of $\mathbb{P}_{\mathrm{int}}$, and the set $\mathbb{B}(\lambda) \cup \{\theta\}$ is stable under the action of all the root operators.*

Hence it follows from Theorem 1.2.3 that the set $\mathbb{B}(\lambda)$ together with the root operators and the maps defined by (1.2.6) has a crystal structure. Furthermore, by Theorem 1.3.1, there is a natural action of the Weyl group W on the set $\mathbb{B}(\lambda)$.

Remark 1.4.6. From the definition of LS paths, it follows that $\pi_\nu := (\nu\,; 0, 1) \in \mathbb{B}(\lambda)$ for all $\nu \in W\lambda$. Hence the $\mathbb{B}(\lambda)$, regarded as a subset of $\mathbb{P}$ by (1.4.1), contains the straight line $\pi_\nu(t) = t\nu$, $t \in [0, 1]$, for all $\nu \in W\lambda$.

Remark 1.4.7. In [L1,L2], Littelmann defined an equivalence relation $\sim$ on the set $\mathbb{P}$ by: $\pi \sim \pi'$ if π is equal to π' modulo reparametrization (see Remark 1.2.1), and equipped the quotient set $\mathbb{P}/\sim$ with a crystal structure by defining root operators on it. Concerning this quotient set $\mathbb{P}/\sim$, a few remarks are in order:

(1) It can be shown that the restriction of the canonical projection $\mathbb{P} \twoheadrightarrow \mathbb{P}/\sim$ to $\mathbb{B}(\lambda)$ is an injective, strict morphism of crystals for every integral weight $\lambda \in P$. Thus, $\mathbb{B}(\lambda)$ can be regarded as a subcrystal of $\mathbb{P}/\sim$.

(2) Similarly, we can regard the set $\mathbb{B}(\varpi_i)^{*m} \subset \mathbb{P}$ defined in Proposition 3.2.1 below as a subcrystal of $\mathbb{P}/\sim$ for each $m \geqslant 1$.

2. Crystal structure of $\mathbb{B}(\varpi_i)$

2.1. Connectedness of $\mathbb{B}(\varpi_i)$

The aim of this subsection is to prove the following theorem.

Theorem 2.1.1. *The crystal graph of the crystal $\mathbb{B}(\varpi_i)$ of all LS paths of shape ϖ_i is connected. In particular, the set $\mathbb{B}_0(\varpi_i)$ in [NS, Theorem 5.1], which is the connected component of $\mathbb{P}$ containing the straight line $\pi_{\varpi_i}(t) := t\varpi_i$, $t \in [0, 1]$, is equal to $\mathbb{B}(\varpi_i)$.*

In order to prove the theorem above, we need some lemmas. The next lemma immediately follows from [AK, Lemma 1.4].

Lemma 2.1.2. *Let $v \in W\varpi_i$. Then, there exist $j_1, j_2, \ldots, j_n \in I$ such that*

(1) $(r_{j_{k-1}} r_{j_{k-2}} \cdots r_{j_1}(v))(h_{j_k}) > 0$ *for all* $k = 1, 2, \ldots, n$;
(2) $r_{j_n} r_{j_{n-1}} \cdots r_{j_1}(v) = \varpi_i + d\delta$ *for some* $d \in \mathbb{Z}$, *where* $\delta \in \mathfrak{h}^*$ *is the null root of* $\mathfrak{g}$.

For $\pi \in \mathbb{B}(\varpi_i)$ and $j \in I$, we set $e_j^{\max}\pi := e_j^{\varepsilon_j(\pi)}\pi$ and $f_j^{\max}\pi := f_j^{\varphi_j(\pi)}\pi$.

Lemma 2.1.3. *Let $\pi = (v_1, v_2, \ldots, v_s\,;\, \underline{a}) \in \mathbb{B}(\varpi_i)$ be an LS path of shape ϖ_i. Assume that $v_s(h_j) > 0$ for some $j \in I$. If we write $f_j^{\max}\pi \in \mathbb{B}(\varpi_i)$ as: $f_j^{\max}\pi = (v'_1, v'_2, \ldots, v'_{s'}\,;\, \underline{a}') \in \mathbb{B}(\varpi_i)$, then we have $v'_{s'} = r_j(v_s)$. Namely, the final direction $v'_{s'}$ of $f_j^{\max}\pi$ is given by $r_j(v_s)$.*

Proof. Let $\pi' := f_j^{\max}\pi$. We deduce from [L2, Proposition 4.7] that the final direction $v'_{s'}$ of π' is equal either to v_s or to $r_j(v_s)$. Suppose that $v'_{s'} = v_s$. Since $v'_{s'}(h_j) = v_s(h_j) > 0$ by assumption, it follows that $H_j^{\pi'}(1) > m_j^{\pi'}$. But, since an LS path satisfies the condition (INT) by Theorem 1.4.5, $H_j^{\pi'}(1) - m_j^{\pi'}$ is a nonnegative integer (see Remark 1.2.2). Therefore, we get $H_j^{\pi'}(1) - m_j^{\pi'} \geqslant 1$, and hence $\varphi_j(\pi') \geqslant 1$ by Theorem 1.2.3. This implies that $f_j\pi' \neq \theta$, which contradicts the definition of $\pi' = f_j^{\max}\pi$. Thus we obtain $v'_{s'} = r_j(v_s)$. $\square$

Lemma 2.1.4. *Let $\lambda \in \mathfrak{h}^*$, and let β, γ be positive real roots. If $r_\beta(\lambda) = r_\gamma(\lambda) \neq \lambda$, then we have $\beta = \gamma$.*

Proof. Since $r_\beta(\lambda) = r_\gamma(\lambda) \neq \lambda$, we see that $\lambda(\beta^\vee)$, $\lambda(\gamma^\vee)$ are nonzero, and $\lambda(\beta^\vee)\beta = \lambda(\gamma^\vee)\gamma$, and hence that $\frac{\lambda(\beta^\vee)}{\lambda(\gamma^\vee)}\beta = \gamma$. But, since β and γ are positive real roots, it follows that $\frac{\lambda(\beta^\vee)}{\lambda(\gamma^\vee)} = 1$, i.e., $\beta = \gamma$. $\square$

Lemma 2.1.5. *Let $\varpi_i + d\delta \in W\varpi_i$ with $d \in \mathbb{Z}$, and let $v \in W\varpi_i$ with $v > \varpi_i + d\delta$. Then, for any rational number $0 < a < 1$, there does not exist an a-chain for $(v, \varpi_i + d\delta)$.*

Proof. Fix a rational number $0 < a < 1$. By the definition of a-chains (see Definition 1.4.3), it suffices to show that there does not exist $\mu \in W\varpi_i$ with $\mu > \varpi_i + d\delta$ such that $\mathrm{dist}(\mu, \varpi_i + d\delta) = 1$ and $a\mu(\beta^\vee) \in \mathbb{Z}$, where β is the positive real root corresponding to the a-chain for $(\mu, \varpi_i + d\delta)$. Suppose that there exists such $\mu \in W\varpi_i$. Then we have $r_\beta(\mu) = \varpi_i + d\delta$ and $(\varpi_i + d\delta)(\beta^\vee) = (r_\beta(\mu))(\beta^\vee) = -\mu(\beta^\vee) > 0$ by the definition of the ordering $\geqslant$ on $W\varpi_i$ (see Definition 1.4.1). In addition, we have $(\varpi_i + d\delta)(\beta^\vee) = (r_\beta(\mu))(\beta^\vee) = -\mu(\beta^\vee) \in a^{-1}\mathbb{Z}$. Combining these, we get

$$(\varpi_i + d\delta)(\beta^\vee) \in a^{-1}\mathbb{Z}_{>0}. \tag{2.1.1}$$

First, we assume that $\mathfrak{g}$ is not of type $A_{2L}^{(2)}$. Then, by [Kac, Proposition 6.3], β is either of the form $\alpha + n\delta$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 0}$, or of the form $-\alpha + n\delta$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 1}$, where $\mathring{\Delta}_+$ is the set of positive roots of the finite-dimensional simple Lie subalgebra $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$ corresponding to the subset $I_0 = I \setminus \{0\}$. But, since $(\varpi_i + d\delta)(\beta^\vee) > 0$ by (2.1.1), the latter case cannot arise. So, it follows that $\beta = \alpha + n\delta$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 0}$. If $n \geqslant 1$, then we deduce by simple computation that

$$\xi_0 := \mu, \quad \xi_1 := r_\alpha(\mu), \quad \xi_2 := r_{-\alpha+n\delta} r_\alpha(\mu), \quad \xi_3 := r_\alpha r_{-\alpha+n\delta} r_\alpha(\mu) = \varpi_i + d\delta$$

is a chain for $(\mu, \varpi_i + d\delta)$, with $\beta_1 := \alpha$, $\beta_2 := -\alpha + n\delta$, $\beta_3 := \alpha$ the corresponding positive real roots (see Definition 1.4.1). This implies that $\mathrm{dist}(\mu, \varpi_i + d\delta) \geqslant 3$, which is a contradiction. Thus we obtain that $n = 0$, and hence $r_\beta = r_\alpha \in \mathring{W}$, where $\mathring{W}$ is the subgroup of W generated by $\{r_j \mid j \in I_0\}$ (i.e., $\mathring{W}$ is the Weyl group of $\mathring{\mathfrak{g}}$). Let $w \in \mathring{W}$ be the shortest element such that

$$\mu = r_\beta(\varpi_i + d\delta) = w(\varpi_i + d\delta)$$

and $w = r_{j_1} r_{j_2} \cdots r_{j_k}$ a reduced expression of $w \in \mathring{W}$ (note that $j_1, j_2, \ldots, j_k \in I_0$, and that $j_k = i$). Since $\varpi_i + d\delta$ is a dominant integral weight with respect to the simple coroots $\{h_j\}_{j \in I_0}$ of $\mathring{\mathfrak{g}}$), we can easily check that

$$\xi_0 := \mu, \quad \xi_1 := r_{j_1}(\mu), \quad \xi_2 := r_{j_2} r_{j_1}(\mu), \quad \ldots, \quad \xi_k := r_{j_k} \cdots r_{j_2} r_{j_1}(\mu) = \varpi_i + d\delta$$

is a chain for $(\mu, \varpi_i + d\delta)$, with $\beta_1 := \alpha_{j_1}$, $\beta_2 := \alpha_{j_2}$, $\ldots$, $\beta_k := \alpha_{j_k}$ the corresponding positive real roots. This implies that $\mathrm{dist}(\mu, \varpi_i + d\delta) \geqslant k$. But, since $\mathrm{dist}(\mu, \varpi_i + d\delta) = 1$

by assumption, we conclude that $k = 1$, and hence $w = r_i$. Then it follows from Lemma 2.1.4 that $\beta = \alpha_i$. Thus, we obtain that $(\varpi_i + d\delta)(\beta^\vee) = (\varpi_i + d\delta)(h_i) = 1 \notin a^{-1}\mathbb{Z}_{>0}$, which contradicts (2.1.1). This proves the lemma in the case where $\mathfrak{g}$ is not of type $A_{2L}^{(2)}$.

Next, we assume that $\mathfrak{g}$ is of type $A_{2L}^{(2)}$. By [Kac, Proposition 6.3] and the inequality (2.1.1), β is either of the form $\alpha + n\delta$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 0}$, or of the form $\frac{1}{2}(\alpha + (2n-1)\delta)$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 1}$. In the same way as above, we can show that the former case cannot arise. So, it follows that $\beta = \frac{1}{2}(\alpha + (2n-1)\delta)$ for some $\alpha \in \mathring{\Delta}_+$ and $n \in \mathbb{Z}_{\geqslant 1}$. Now we see easily that

$$\xi_0 := \mu, \quad \xi_1 := r_\alpha(\mu), \quad \xi_2 := r_\gamma r_\alpha(\mu), \quad \xi_3 := r_\alpha r_\gamma r_\alpha(\mu) = \varpi_i + d\delta,$$

is a chain for $(\mu, \varpi_i + d\delta)$, where $\gamma := \frac{1}{2}(-\alpha + (2n-1)\delta)$. This implies that $\mathrm{dist}(\mu, \varpi_i + d\delta) \geqslant 3$, which is again a contradiction. Thus we have proved the lemma. $\square$

Proof of Theorem 2.1.1. It suffices to prove that every element $\pi \in \mathbb{B}(\varpi_i)$ is connected to the straight line $\pi_{\varpi_i} = (\varpi_i\,;\,0,1)$ in the crystal graph (see Remark 1.4.6). Let us write $\pi \in \mathbb{B}(\varpi_i)$ as: $\pi = (v_1, v_2, \ldots, v_s\,;\,\underline{a})$. By Lemma 2.1.2, there exist $j_1, j_2, \ldots, j_n \in I$ such that

(1) $(r_{j_{k-1}} r_{j_{k-2}} \cdots r_{j_1}(v_s))(h_{j_k}) > 0$ for all $k = 1, 2, \ldots, n$;
(2) $r_{j_n} r_{j_{n-1}} \cdots r_{j_1}(v_s) = \varpi_i + d\delta$ for some $d \in \mathbb{Z}$.

We set $\pi' := f_{j_n}^{\max} f_{j_{n-1}}^{\max} \cdots f_{j_1}^{\max}\pi$. By successive applications of Lemma 2.1.3, we deduce that the final direction of π' is equal to $r_{j_n} r_{j_{n-1}} \cdots r_{j_1}(v_s) = \varpi_i + d\delta$. Then it follows from Lemma 2.1.5 and the definition of LS paths (see Definition 1.4.4) that π' is the straight line $(\varpi_i + d\delta\,;\,0,1)$. Note that there exists $w \in W$ such that $w(\varpi_i + d\delta) = \varpi_i$, since $\varpi_i + d\delta \in W\varpi_i$. Then, using [GL, Lemma 5.2], we deduce that $S_w\pi' = \pi_{\varpi_i}$, which completes the proof of the theorem. $\square$

2.2. Relation to the crystal base $\mathcal{B}(\varpi_i)$

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of the affine Lie algebra $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q. For an integral weight $\lambda \in P$, we denote by $V(\lambda)$ the extremal weight module over $U_q(\mathfrak{g})$ of extremal weight λ, which is an integrable $U_q(\mathfrak{g})$-module introduced by Kashiwara [Kas1, §8] as a natural generalization of an integrable highest (or lowest) weight $U_q(\mathfrak{g})$-module. We know from [Kas1, Proposition 8.2.2] that the extremal weight module $V(\lambda)$ has a crystal base, denoted by $\mathcal{B}(\lambda)$.

Now, by combining Theorem 2.1.1 with the main result of our previous work [NS, Theorem 5.1], we obtain the following:

Corollary 2.2.1. *The crystal $\mathbb{B}(\varpi_i)$ of all LS paths of shape ϖ_i is, as a crystal, isomorphic to the crystal base $\mathcal{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight ϖ_i.*

3. Crystal structure of $\mathbb{B}(m\varpi_i)$

3.1. Main result

Let $\mathrm{Par}_{<m}$ be the set of all partitions of length (i.e., number of parts) strictly less than m. For each $\sigma \in \mathrm{Par}_{<m}$ of the form $(k_1, k_2, \ldots, k_{m-1})$ with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$, we denote by $|\sigma|$ the weight of σ, i.e., $|\sigma| := k_1 + k_2 + \cdots + k_{m-1}$. We can define a crystal structure on $\mathrm{Par}_{<m}$ as follows:

$$\begin{cases} e_j\sigma = f_j\sigma = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \varepsilon_j(\sigma) = \varphi_j(\sigma) = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \mathrm{wt}(\sigma) = -|\sigma| d_i \delta & \text{for } \sigma \in \mathrm{Par}_{<m}, \end{cases}$$

where the $0\,(= e_j\sigma = f_j\sigma)$ is an extra element, which is not contained in $\mathrm{Par}_{<m}$, and d_i is a positive integer determined by:

$$\left\{ d \in \mathbb{Z} \mid \varpi_i + d\delta \in W\varpi_i \right\} = \mathbb{Z} d_i. \tag{3.1.1}$$

Our main result of this paper is the following:

Theorem 3.1.1. *The crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the tensor product $\mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i)$ of the crystals $\mathrm{Par}_{<m}$ and $\mathbb{B}_0(m\varpi_i)$, where $\mathbb{B}_0(m\varpi_i)$ is the connected component of $\mathbb{B}(m\varpi_i)$ containing the straight line $\pi_{m\varpi_i}(t) := t(m\varpi_i)$, $t \in [0, 1]$.*

The rest of this section is devoted to the proof of Theorem 3.1.1.

3.2. Concatenations

Fix a positive integer $m \in \mathbb{Z}$. For m paths $\pi_1, \pi_2, \ldots, \pi_m \in \mathbb{P}$, we define a concatenation $\pi_1 * \pi_2 * \cdots * \pi_m \in \mathbb{P}$ of them by (cf. [L2, §1]):

$$(\pi_1 * \pi_2 * \cdots * \pi_m)(t) = \sum_{l=1}^{k-1} \pi_l(1) + \pi_k(mt - k + 1)$$

$$\text{for} \quad \frac{k-1}{m} \leqslant t \leqslant \frac{k}{m}, \quad 1 \leqslant k \leqslant m. \tag{3.2.1}$$

The proof of the following proposition can easily be reduced to that of [L2, Lemma 2.7], where the case $m = 2$ is treated.

Proposition 3.2.1. *We set*

$$\mathbb{B}(\varpi_i)^{*m} := \left\{ \pi_1 * \pi_2 * \cdots * \pi_m \mid \pi_k \in \mathbb{B}(\varpi_i) \;\; for \;\; k = 1, 2, \ldots, m \right\}. \tag{3.2.2}$$

Then, $\mathbb{B}(\varpi_i)^{*m}$ is a subset of $\mathbb{P}_{\mathrm{int}}$, and the set $\mathbb{B}(\varpi_i)^{*m} \cup \{\theta\}$ is stable under the action of all the root operators.

Proposition 3.2.2. *The set $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is a subset (and hence a subcrystal) of $\mathbb{B}(\varpi_i)^{*m}$ consisting of elements of the form $\pi_1 * \pi_2 * \cdots * \pi_m$ satisfying the following condition: $\nu_{k,s_k} \geqslant \nu_{k+1,1}$ in $W\varpi_i$ for each $k = 1, 2, \ldots, m-1$, where we write $\pi_k \in \mathbb{B}(\varpi_i)$ as: $\pi_k = (\nu_{k,1}, \nu_{k,2}, \ldots, \nu_{k,s_k}; a_{k,0}, a_{k,1}, \ldots, a_{k,s_k})$ for $k = 1, 2, \ldots, m$.*

In order to prove the proposition above, we need some lemmas.

Lemma 3.2.3. *Let $\mu, \nu \in W\varpi_i$. Then, $m\mu > m\nu$ in $W(m\varpi_i)$ and $\mathrm{dist}(m\mu, m\nu) = 1$ if and only if $\mu > \nu$ in $W\varpi_i$ and $\mathrm{dist}(\mu, \nu) = 1$.*

Proof. We see by simple computation that $\mu = \xi_0, \xi_1, \ldots, \xi_r = \nu$ is a chain for (μ, ν) with the corresponding positive real roots $\beta_1, \beta_2, \ldots, \beta_r$ if and only if $m\mu = m\xi_0, m\xi_1, \ldots, m\xi_r = m\nu$ is a chain for $(m\mu, m\nu)$ with the corresponding positive real roots $\beta_1, \beta_2, \ldots, \beta_r$. Hence the assertion of the lemma immediately follows. □

Lemma 3.2.4. *Let $\mu, \nu \in W\varpi_i$, and let $\frac{k-1}{m} < a < \frac{k}{m}$ be a rational number, where k is a fixed integer between 1 and m. Assume that $m\mu > m\nu$ in $W(m\varpi_i)$, and that there exists an a-chain for $(m\mu, m\nu)$. Then there exists an $(ma - k + 1)$-chain for (μ, ν).*

Proof. Let $m\mu = m\xi_0 > m\xi_1 > \cdots > m\xi_r = m\nu$ be an a-chain for $(m\mu, m\nu)$, where $\xi_l \in W\varpi_i$ for $l = 1, 2, \ldots, r$, and let $\beta_1, \beta_2, \ldots, \beta_r$ be the positive real roots corresponding to this a-chain. Then we deduce from Lemma 3.2.3 that

$$\mu = \xi_0 > \xi_1 > \cdots > \xi_r = \nu \tag{3.2.3}$$

in $W\varpi_i$, and $\mathrm{dist}(\xi_{l-1}, \xi_l) = 1$ for all $l = 1, 2, \ldots, r$. But, since $a(m\xi_{l-1})(\beta_l^\vee) \in \mathbb{Z}$, and $\xi_{l-1} \in W\varpi_i$ is an integral weight, we have

$$(ma - k + 1)\xi_{l-1}(\beta_l^\vee) = a(m\xi_{l-1}(\beta_l^\vee)) - (k-1)\xi_{l-1}(\beta_l^\vee) \in \mathbb{Z}$$

for all $l = 1, 2, \ldots, r$. Hence it follows that sequence (3.2.3) is an $(ma-k+1)$-chain for (μ, ν), with $\beta_1, \beta_2, \ldots, \beta_r$ the corresponding positive real roots. This proves the lemma. □

We can prove the following lemma in the same way as lemmas above.

Lemma 3.2.5. (1) *Let $\mu, \nu \in W\varpi_i$, and let $0 < a < 1$ be a rational number. If there exists an a-chain for (μ, ν), then for each $1 \leqslant k \leqslant m-1$, there exists some $\left(\frac{a+k}{m}\right)$-chain for $(m\mu, m\nu)$.*

(2) *Let $\mu, \nu \in W\varpi_i$ with $\mu > \nu$. For each $1 \leqslant k \leqslant m-1$, there exists some $\left(\frac{k}{m}\right)$-chain for $(m\mu, m\nu)$.*

Proof of Proposition 3.2.2. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s\,;\, a_0, a_1, \ldots, a_s) \in \mathbb{B}(m\varpi_i)$. We "divide" the path π into m paths π_k, $k = 1, 2, \ldots, m$, as follows: $\pi_k(t) = \pi(\frac{t+k-1}{m}) - \pi(\frac{k-1}{m})$, $t \in [0, 1]$. It readily follows from definition (3.2.1) that $\pi = \pi_1 * \pi_2 * \cdots * \pi_m$.

We will show that each π_k is an LS path of shape ϖ_i for $k = 1, 2, \ldots, m$. For simplicity, we give a proof only for the case where $m = 2$ (the proof for the general case is similar). If there exists $0 \leqslant s_1 \leqslant s$ such that $a_{s_1} = \frac{1}{2}$, then π_1 and π_2 are given as:

$$\pi_1 = \left(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2, \ldots, \frac{1}{2}\nu_{s_1}\,;\, 2a_0, 2a_1, \ldots, 2a_{s_1}\right),$$

$$\pi_2 = \left(\frac{1}{2}\nu_{s_1+1}, \frac{1}{2}\nu_{s_1+2}, \ldots, \frac{1}{2}\nu_s\,;\, 2a_{s_1} - 1, 2a_{s_1+1} - 1, \ldots, 2a_s - 1\right).$$

If $a_{s_1-1} < \frac{1}{2} < a_{s_1}$ for some $1 \leqslant s_1 \leqslant s$, then π_1 and π_2 are given as:

$$\pi_1 = \left(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2, \ldots, \frac{1}{2}\nu_{s_1}\,;\, 2a_0, 2a_1, \ldots, 2a_{s_1-1}, 1\right),$$

$$\pi_2 = \left(\frac{1}{2}\nu_{s_1}, \frac{1}{2}\nu_{s_1+1}, \frac{1}{2}\nu_{s_1+2}, \ldots, \frac{1}{2}\nu_s\,;\, 0, 2a_{s_1} - 1, 2a_{s_1+1} - 1, \ldots, 2a_s - 1\right).$$

In both cases, we deduce from Lemmas 3.2.3 and 3.2.4 that π_1 and π_2 are LS paths of shape ϖ_i, and that the final direction of π_1 is greater than or equal to the initial direction of π_2.

Conversely, let $\pi_k = (\nu_{k,1}, \nu_{k,2}, \ldots, \nu_{k,s_k}\,;\, a_{k,0}, a_{k,1}, \ldots, a_{k,s_k})$, $k = 1, 2, \ldots, m$, be elements of $\mathbb{B}(\varpi_i)$ such that $\nu_{k,s_k} \geqslant \nu_{k+1,1}$ for each $k = 1, 2, \ldots, m-1$. We will show that the path $\pi := \pi_1 * \pi_2 * \cdots * \pi_m$ is an LS path of shape $m\varpi_i$. As above, we give a proof only for the case where $m = 2$. Then, the path $\pi = \pi_1 * \pi_2$ is given as follows: If $\nu_{1,s_1} = \nu_{2,1}$, then

$$\pi = \Bigg(2\nu_{1,1}, \ldots, 2\nu_{1,s_1}, 2\nu_{2,2}, \ldots, 2\nu_{2,s_2}\,;\, \frac{a_{1,0}}{2}, \ldots, \frac{a_{1,s_1-1}}{2}, \frac{a_{2,1}+1}{2}, \ldots, \frac{a_{2,s_2}+1}{2}\Bigg).$$

If $\nu_{1,s_1} > \nu_{2,1}$, then

$$\pi = \Bigg(2\nu_{1,1}, \ldots, 2\nu_{1,s_1}, 2\nu_{2,1}, \ldots, 2\nu_{2,s_2}\,;\, \frac{a_{1,0}}{2}, \ldots, \frac{a_{1,s_1}}{2}, \frac{a_{2,1}+1}{2}, \ldots, \frac{a_{2,s_2}+1}{2}\Bigg).$$

In both cases, we deduce from Lemmas 3.2.3 and 3.2.5 that π is an LS path of shape $2\varpi_i$. This completes the proof of the proposition. □

Lemma 3.2.6. *Let $k_1, k_2, \ldots, k_m \in \mathbb{Z}$. The path $\pi_{\varpi_i - k_1 d_i \delta} * \pi_{\varpi_i - k_2 d_i \delta} * \cdots * \pi_{\varpi_i - k_m d_i \delta} \in \mathbb{B}(\varpi_i)^{*m}$ is contained in $\mathbb{B}(m\varpi_i)$ if and only if $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$, where $\pi_\nu(t) := t\nu$, $t \in [0, 1]$, for each integral weight $\nu \in P$.*

Proof. Here we remark that $\pi_{\varpi_i - kd_i\delta} = (\varpi_i - kd_i\delta;\, 0, 1) \in \mathbb{B}(\varpi_i)$ for all $k \in \mathbb{Z}$, since $\varpi_i - kd_i\delta \in W\varpi_i$ by the definition of d_i (see Remark 1.4.6). Therefore, by Proposition 3.2.2, it suffices to show that $\varpi_i - pd_i\delta \geqslant \varpi_i - qd_i\delta$ if and only if $p \geqslant q$. It follows from Definition 1.4.1 that $\nu - \mu \in Q_+ := \sum_{j\in I} \mathbb{Z}_{\geqslant 0}\alpha_j$ if $\mu \geqslant \nu$. Hence it is obvious that $\varpi_i - pd_i\delta \geqslant \varpi_i - qd_i\delta$ implies $p \geqslant q$.

Let us show the converse by induction on $p - q$. If $p - q = 0$, then the assertion is obvious. So, we assume that $p - q > 0$. Suppose first that $\mathfrak{g}$ is not of type $A_{2L}^{(2)}$. By [Kas3, (4.10)] and the comment just after [Kas3, (5.6)], $\beta := -\alpha_i + d_i\delta$ and $\gamma := \alpha_i$ are positive real roots. Then we can easily check that

$$\varpi_i - pd_i\delta, \quad r_\beta(\varpi_i - pd_i\delta), \quad r_\gamma r_\beta(\varpi_i - pd_i\delta) = \varpi_i - (p-1)d_i\delta$$

is a chain for $(\varpi_i - pd_i\delta, \varpi_i - (p-1)d_i\delta)$. Hence we have $\varpi_i - pd_i\delta > \varpi_i - (p-1)d_i\delta$. But, by the inductive assumption, we have $\varpi_i - (p-1)d_i\delta > \varpi_i - qd_i\delta$. Combining these, we obtain $\varpi_i - pd_i\delta > \varpi_i - qd_i\delta$. Next we suppose that $\mathfrak{g}$ is of type $A_{2L}^{(2)}$. If the simple root α_i (corresponding to ϖ_i) is not the longest simple root of $\mathfrak{g}$ (i.e., if $i \neq \ell$ in the notation of [Kac, Chapter 4, Table Aff 2], with "L" replacing "ℓ"), then the assertion can be shown in exactly the same way as above. If α_i is the longest simple root (i.e., if $i = L$), then we know from the comment just after [Kas3, (5.6)] that $d_i = 1$, and from [Kac, Proposition 6.3] that $\beta := \frac{1}{2}(-\alpha_i + \delta)$ and $\gamma := \alpha_i$ are positive real roots. Then we see that

$$\varpi_i - p\delta, \quad r_\beta(\varpi_i - p\delta), \quad r_\gamma r_\beta(\varpi_i - p\delta) = \varpi_i - (p-1)\delta$$

is a chain for $(\varpi_i - p\delta, \varpi_i - (p-1)\delta)$, and hence that $\varpi_i - p\delta > \varpi_i - (p-1)\delta$. Now, using the inductive assumption as above, we obtain $\varpi_i - p\delta > \varpi_i - q\delta$. This proves the lemma. □

For a partition σ in $\mathrm{Par}_{<m}$ of the form $(k_1, k_2, \ldots, k_{m-1})$ with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$, we denote by $\mathbb{B}_\sigma(m\varpi_i)$ the connected component of $\mathbb{B}(\varpi_i)^{*m}$ containing the path $\pi_{\varpi_i - k_1 d_i\delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}$. The next proposition immediately follows from Lemma 3.2.6.

Proposition 3.2.7. *The connected component $\mathbb{B}_\sigma(m\varpi_i)$ is contained in $\mathbb{B}(m\varpi_i)$ for all $\sigma \in \mathrm{Par}_{<m}$.*

3.3. Path models with weights in P_{cl}

We can easily verify the following lemma by using the definition of root operators.

Lemma 3.3.1. *Let $\pi, \pi' \in \mathbb{P}$. Assume that there exists a piecewise linear, continuous function $F : [0, 1] \to \mathbb{Q}$ with $F(0) = 0$ such that $\pi(t) = \pi'(t) + F(t)\delta$ for all $t \in [0, 1]$. Then, we have for all $j \in I$ and $t \in [0, 1]$,*

$$(e_j\pi)(t) = (e_j\pi')(t) + F(t)\delta \quad \textit{and} \quad (f_j\pi)(t) = (f_j\pi')(t) + F(t)\delta.$$

Here, $\theta + F(t)\delta$ is understood to be θ for $t \in [0, 1]$.

Now, we recall the notion of a $U_q'(\mathfrak{g})$-crystal from [AK, §1], [Kas3, §4]. We set $P_{\mathrm{cl}} := P/(P \cap \mathbb{Q}\delta)$. Let $U_q'(\mathfrak{g})$ be the quantized universal enveloping algebra of the affine Lie algebra $\mathfrak{g}$ over $\mathbb{Q}(q)$ with P_{cl} as an integral weight lattice (see [Kas3, §4.2]). We call a crystal with P_{cl} as an integral weight lattice a $U_q'(\mathfrak{g})$-crystal (for the precise definition of a $U_q'(\mathfrak{g})$-crystal, see [Kas2, §7.2], where "P" should read "P_{cl}"). Also, a $U_q'(\mathfrak{g})$-crystal $\mathcal{B}$ is called a regular crystal if for every proper subset J of I, $\mathcal{B}$ is isomorphic to the crystal base of an integrable $U_q'(\mathfrak{g}_J)$-module, where $U_q'(\mathfrak{g}_J)$ is the subalgebra of $U_q'(\mathfrak{g})$ corresponding to the subset J of I (see [AK, §1.4]). Furthermore, a regular crystal is said to be simple if it satisfies some additional conditions, for which we refer the reader to [AK, Definition 1.7].

Denote by

$$\mathrm{cl} : \mathbb{Q} \otimes_{\mathbb{Z}} P \twoheadrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}} \tag{3.3.1}$$

the canonical projection. For each $\pi \in \mathbb{P}$, we define a piecewise linear, continuous map $\mathrm{cl}(\pi) : [0, 1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ by $(\mathrm{cl}(\pi))(t) := \mathrm{cl}(\pi(t))$ for $t \in [0, 1]$. Let $\mathbb{B}$ be a subset of $\mathbb{P}_{\mathrm{int}}$ such that the set $\mathbb{B} \cup \{\theta\}$ is stable under all the root operators, such as $\mathbb{B}(m\varpi_i)$ and $\mathbb{B}(\varpi_i)^{*m}$, and set $\mathbb{B}_{\mathrm{cl}} := \mathrm{cl}(\mathbb{B})$. We can endow $\mathbb{B}_{\mathrm{cl}}$ with a $U_q'(\mathfrak{g})$-crystal structure as follows: for $\mathrm{cl}(\pi) \in \mathbb{B}_{\mathrm{cl}}$ and $j \in I$, we define

$$\begin{cases} e_j\,\mathrm{cl}(\pi) := \mathrm{cl}(e_j\pi), \quad f_j\,\mathrm{cl}(\pi) := \mathrm{cl}(f_j\pi), \\ \varepsilon_j(\mathrm{cl}(\pi)) := \varepsilon_j(\pi), \quad \varphi_j(\mathrm{cl}(\pi)) := \varphi_j(\pi), \\ \mathrm{wt}(\mathrm{cl}(\pi)) := \mathrm{cl}(\mathrm{wt}(\pi)). \end{cases} \tag{3.3.2}$$

Here we set $\mathrm{cl}(\theta) := \theta$. Using Lemma 3.3.1, we can check that these maps are all well-defined, and that the set $\mathbb{B}_{\mathrm{cl}}$ together with these maps is a $U_q'(\mathfrak{g})$-crystal.

3.4. Regularity of $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$

In [NS, §5.3], we defined a bijection $z_i : \mathbb{B}(\varpi_i) \to \mathbb{B}(\varpi_i)$ by: $(z_i(\pi))(t) = \pi(t) + td_i\delta$ for $t \in [0, 1]$, and introduced an equivalence relation $\sim$ on $\mathbb{B}(\varpi_i)$ by: $\pi \sim \pi'$ if $\pi =$

$z_i^k(\pi')$ for some $k \in \mathbb{Z}$. We know the following from [NS, Proposition 5.8], [Kas3, Theorem 5.17 (ii)].

Proposition 3.4.1. *We set $\mathbb{B}'(\varpi_i) := \mathbb{B}(\varpi_i)/\sim$. Then, the quotient set $\mathbb{B}'(\varpi_i)$ has a $U_q'(\mathfrak{g})$-crystal structure induced from the crystal structure of $\mathbb{B}(\varpi_i)$. Moreover, $\mathbb{B}'(\varpi_i)$ is, as a $U_q'(\mathfrak{g})$-crystal, isomorphic to the crystal base of a level-zero fundamental $U_q'(\mathfrak{g})$-module. In particular, $\mathbb{B}'(\varpi_i)$ is a simple crystal.*

As mentioned in §3.3, the set $\mathbb{B}(\varpi_i)_{\mathrm{cl}} := \mathrm{cl}(\mathbb{B}(\varpi_i))$ is endowed with a $U_q'(\mathfrak{g})$-crystal structure. Notice that we have a map $q : \mathbb{B}'(\varpi_i) \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ satisfying $q \circ p = \mathrm{cl}$, where $p : \mathbb{B}(\varpi_i) \twoheadrightarrow \mathbb{B}'(\varpi_i)$ is the canonical projection.

Proposition 3.4.2. *The map $q : \mathbb{B}'(\varpi_i) \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is an isomorphism between the $U_q'(\mathfrak{g})$-crystals $\mathbb{B}'(\varpi_i)$ and $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$. In particular, $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is also a simple crystal.*

Proof. From the definitions of $U_q'(\mathfrak{g})$-crystal structures of $\mathbb{B}'(\varpi_i)$ and $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$, it is easily shown that $q : \mathbb{B}'(\varpi_i) \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is a strict morphism (see [Kas1, §1.5]; we set $q(\theta) := \theta$). For example, let us check that $q(e_j b) = e_j q(b)$ for all $b \in \mathbb{B}'(\varpi_i)$ and $j \in I$. Let $\pi \in \mathbb{B}(\varpi_i)$ be a representative of b (i.e., $p(\pi) = b$). Then, since $q \circ p = \mathrm{cl}$, we have $\mathrm{cl}(\pi) = q(b)$. Hence it follows that

$$\begin{aligned} q(e_j b) &= q(p(e_j \pi)) \quad \text{by the definition of } e_j \text{ on } \mathbb{B}'(\varpi_i) \\ &= \mathrm{cl}(e_j \pi) \quad \text{since } q \circ p = \mathrm{cl} \\ &= e_j q(b) \quad \text{by the definition of } e_j \text{ on } \mathbb{B}(\varpi_i)_{\mathrm{cl}}. \end{aligned}$$

Now it remains to prove that $q : \mathbb{B}'(\varpi_i) \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is bijective. It is obvious by the definition that $q : \mathbb{B}'(\varpi_i) \to \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is surjective. Suppose that q is not injective. Then there exist $\pi, \pi' \in \mathbb{B}(\varpi_i)$ such that $p(\pi) \neq p(\pi')$ and $\mathrm{cl}(\pi) = \mathrm{cl}(\pi')$. Since $\mathrm{cl}(\pi) = \mathrm{cl}(\pi')$, we deduce that $\pi(t) = \pi'(t) + F(t)\delta$ for some piecewise linear, continuous function $F : [0, 1] \to \mathbb{Q}$. Suppose that the path $F(t)\delta$, $t \in [0, 1]$, is a straight line, i.e., that $F(t) = tR$, $t \in [0, 1]$, for some $R \in \mathbb{Q}$. Because the crystal graph of $\mathbb{B}(\varpi_i)$ is connected by Theorem 2.1.1, there exist $j_1, j_2, \ldots, j_n \in I$ such that $x_{j_1} x_{j_2} \cdots x_{j_n} \pi' = (\varpi_i\,;\, 0, 1)$, where x_j is either e_j or f_j. Then, by successively applying Lemma 3.3.1, we obtain

$$(x_{j_1} x_{j_2} \cdots x_{j_n} \pi)(t) = (x_{j_1} x_{j_2} \cdots x_{j_n} \pi')(t) + F(t)\delta \tag{3.4.1}$$

and hence $x_{j_1} x_{j_2} \cdots x_{j_n} \pi = (\varpi_i + R\delta\,;\, 0, 1)$. But, since $x_{j_1} x_{j_2} \cdots x_{j_n} \pi \in \mathbb{B}(\varpi_i)$ by Theorem 1.4.5, we have $\varpi_i + R\delta \in W\varpi_i$. Hence $R \in \mathbb{Z}$, and $R = R'd_i$ for some $R' \in \mathbb{Z}$. So, we have $\pi(t) = \pi'(t) + tR'd_i\delta$, $t \in [0, 1]$. This implies that $\pi \sim \pi'$, which contradicts the assumption that $p(\pi) \neq p(\pi')$. Thus, the path $F(t)\delta$, $t \in [0, 1]$, is not a straight line.

Let us write $F(t)$ as:

$$F(t) = \sum_{l=1}^{k-1} (a_l - a_{l-1}) R_l + (t - a_{k-1}) R_k \quad \text{for } t \in [a_{k-1}, a_k],\ 1 \leqslant k \leqslant s,$$

for some $0 = a_0 < a_1 < a_2 < \cdots < a_s = 1$ and some $R_1, R_2, \ldots, R_s \in \mathbb{Q}$ satisfying $R_k \neq R_{k+1}$ for any $k = 1, 2, \ldots, s-1$ (note that $s \geqslant 2$). Then, by (3.4.1), we have

$$x_{j_1} x_{j_2} \cdots x_{j_n} \pi = (\varpi_i + R_1\delta,\ \varpi_i + R_2\delta,\ \ldots,\ \varpi_i + R_s\delta\,;\ a_0, a_1, \ldots, a_s).$$

Since $x_{j_1} x_{j_2} \cdots x_{j_n} \pi \in \mathbb{B}(\varpi_i)$, it follows from the definition of LS paths that $R_k \in \mathbb{Z}$, $R_k \in \mathbb{Z} d_i$ for all $k = 1, 2, \ldots, s$, and that there exists an a_1-chain for $(\varpi_i + R_1\delta,\ \varpi_i + R_2\delta)$. This contradicts Lemma 2.1.5. Thus we have proved the proposition. $\square$

Lemma 3.4.3. *For each* $m \geqslant 1$, $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{*m} := \mathrm{cl}(\mathbb{B}(\varpi_i)^{*m})$ *is, as a* $U_q'(\mathfrak{g})$*-crystal, isomorphic to the m-fold tensor product* $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m} := \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ (m-times) *of* $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$.

Proof. First we remark that $\mathrm{cl}(\pi_1 * \pi_2 * \cdots * \pi_m) = \mathrm{cl}(\pi_1) * \mathrm{cl}(\pi_2) * \cdots * \mathrm{cl}(\pi_m)$ for $\pi_1, \pi_2, \ldots, \pi_m \in \mathbb{P}$, where the concatenation $\mathrm{cl}(\pi_1) * \mathrm{cl}(\pi_2) * \cdots * \mathrm{cl}(\pi_m)$ of the "paths" $\mathrm{cl}(\pi_1), \mathrm{cl}(\pi_2), \ldots, \mathrm{cl}(\pi_m) : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ is defined as in (3.2.1). Then, the assertion of the lemma immediately follows from the definition (3.3.2) of the $U_q'(\mathfrak{g})$-crystal structure of $\mathbb{B}(\varpi_i)_{\mathrm{cl}} = \mathrm{cl}(\mathbb{B}(\varpi_i))$ and the fact that concatenations correspond to tensor products also for this crystal. $\square$

Proposition 3.4.4. *For each* $m \geqslant 1$, $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ *is isomorphic to the tensor product* $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ *as a* $U_q'(\mathfrak{g})$*-crystal. In particular,* $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ *is simple, and hence regular.*

Proof. We deduce from Proposition 3.2.2 that $\mathbb{B}(m\varpi_i)_{\mathrm{cl}} := \mathrm{cl}(\mathbb{B}(m\varpi_i))$ is a subcrystal of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{*m} := \mathrm{cl}(\mathbb{B}(\varpi_i)^{*m})$, and from Lemma 3.4.3 that $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{*m}$ is isomorphic to the tensor product $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$. Hence $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ is (isomorphic to) a subcrystal of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$. On the other hand, by [AK, Lemma 1.10] and Proposition 3.4.2, we see that $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ is simple. Since the crystal graph of a simple crystal is connected (see [AK, Lemma 1.9]), and since $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ is (isomorphic to) a subcrystal of the simple crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$, we conclude that $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ is isomorphic to the simple crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ as a $U_q'(\mathfrak{g})$-crystal. This proves the proposition. $\square$

3.5. Extremal elements in $\mathbb{B}(m\varpi_i)$

Recall that there is a natural action of the Weyl group W on a path crystal (see Theorem 1.3.1). In a similar way, we can define an action of the Weyl group W on a regular ($U_q'(\mathfrak{g})$)-crystal. Let $\mathcal{B}$ be a regular ($U_q'(\mathfrak{g})$)-crystal. For each $j \in I$, we define

$S_j : \mathcal{B} \to \mathcal{B}$ by:

$$S_j b = \begin{cases} f_j^n b & \text{if } \ n := (\mathrm{wt}\, b)(h_j) \geqslant 0 \\ e_j^{-n} b & \text{if } \ n := (\mathrm{wt}\, b)(h_j) < 0 \end{cases} \qquad \text{for } \ b \in \mathcal{B}.$$

Here we set $(\mathrm{cl}(\lambda))(h_j) := \lambda(h_j)$ for $\lambda \in P$ and $j \in I$; this is well-defined, since $\delta(h_j) = 0$ for all $j \in I$. Note that we have an action of the Weyl group W on P_{cl} induced from that on P, since $W\delta = \delta$.

Theorem 3.5.1 (*Kashiwara [Kas1, §7]*). *Let $\mathcal{B}$ be a regular crystal. Then, there exists a unique action $S : W \to \mathrm{Bij}(\mathcal{B})$, $w \mapsto S_w$, of the Weyl group W on the set $\mathcal{B}$ such that $S_{r_j} = S_j$ for all $j \in I$, where $\mathrm{Bij}(\mathcal{B})$ is the group of all bijections from the set $\mathcal{B}$ to itself. In addition, we have $\mathrm{wt}(S_w b) = w(\mathrm{wt}(b))$ for $w \in W$ and $b \in \mathcal{B}$.*

Definition 3.5.2. Let $\mathcal{B}$ be either a subset of $\mathbb{P}_{\mathrm{int}}$ such that the set $\mathcal{B} \cup \{\theta\}$ is stable under all the root operators, or a regular $(U_q'(\mathfrak{g}))$-crystal. An element $b \in \mathcal{B}$ is said to be extremal, if for every $w \in W$, we have $e_j S_w b = \theta$ or $f_j S_w b = \theta$ for each $j \in I$.

The next lemma immediately follows from definition (3.3.2) of the $U_q'(\mathfrak{g})$-crystal structure of $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$.

Lemma 3.5.3. *We have $\mathrm{cl}(S_w \pi) = S_w(\mathrm{cl}(\pi))$ for all $w \in W$ and $\pi \in \mathbb{B}(m\varpi_i)$. Moreover, an element $\pi \in \mathbb{B}(m\varpi_i)$ is extremal if and only if $\mathrm{cl}(\pi) \in \mathbb{B}(m\varpi_i)_{\mathrm{cl}} \cong \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ is extremal.*

Lemma 3.5.4. (1) *The path $\pi_{\varpi_i - k_1 d_i \delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}$ is an extremal element of $\mathbb{B}(m\varpi_i)$ for all $k_1, k_2, \ldots, k_{m-1} \in \mathbb{Z}$ such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$.*

(2) *Each connected component of $\mathbb{B}(m\varpi_i)$ contains an extremal element.*

(3) *Let $\pi \in \mathbb{B}(m\varpi_i)$ be an extremal element. Then, π is in the Weyl group orbit of $\pi_{\varpi_i - k_1 d_i \delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}$ for some $k_1, k_2, \ldots, k_{m-1} \in \mathbb{Z}$ such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$.*

Proof. (1) By induction on the length of $w \in W$, we can show that

$$S_w(\pi_{\varpi_i - k_1 d_i \delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}) = \pi_{w(\varpi_i) - k_1 d_i \delta} * \cdots * \pi_{w(\varpi_i) - k_{m-1} d_i \delta} * \pi_{w(\varpi_i)}$$

for all $w \in W$ (cf. [GL, Lemma 5.2]). Now, part (1) immediately follows from this equality and the definition of root operators.

(2) Let $\pi \in \mathbb{B}(m\varpi_i)$. Then it follows from Proposition 3.4.4 and [AK, Lemma 1.5] that $e_{j_1} e_{j_2} \cdots e_{j_n} \mathrm{cl}(\pi)$ is extremal for some $j_1, j_2, \ldots, j_n \in I$. But, by definition (3.3.2), we have

$$e_{j_1} e_{j_2} \cdots e_{j_n} \mathrm{cl}(\pi) = \mathrm{cl}(e_{j_1} e_{j_2} \cdots e_{j_n} \pi).$$

Hence, by Lemma 3.5.3, $e_{j_1} e_{j_2} \cdots e_{j_n} \pi$ is also extremal. This proves part (2).

(3) Recall that there is a natural action of the Weyl group W on the regular crystal $\mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ (see Theorem 3.5.1). We first prove the following:

Claim. *Let $b := b_1 \otimes b_2 \otimes \cdots \otimes b_m$ be an extremal element of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ with $b_1, b_2, \ldots, b_m \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$. If the weight $\mathrm{wt}(b) \in P_{\mathrm{cl}}$ of b is contained in the dominant Weyl chamber $\mathring{C} := \{\lambda \in P_{\mathrm{cl}} \mid \lambda(h_j) \geqslant 0$ for all $j \in I_0\}$ with respect to the simple roots $\{\alpha_j\}_{j \in I_0}$ of $\mathring{\mathfrak{g}}$, then we have $b = \mathrm{cl}(\pi_{\varpi_i})^{\otimes m}$. Here, $\mathring{\mathfrak{g}}$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to the subset $I_0 = I \setminus \{0\}$, which is a finite-dimensional simple Lie algebra.*

Proof of the claim. Here we remark that every extremal element of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is in the Weyl group orbit of $\mathrm{cl}(\pi_{\varpi_i})$. Indeed, if $\mathrm{cl}(\pi) \in \mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is extremal for $\pi \in \mathbb{B}(\varpi_i)$, then by Lemma 3.5.3 (for the case $m = 1$), π is also an extremal element of $\mathbb{B}(\varpi_i)$. But, because $\mathbb{B}(\varpi_i)$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight ϖ_i by Corollary 2.2.1, we deduce from [Kas3, Theorem 5.5] that π is in the Weyl group orbit of π_{ϖ_i}, i.e., that $\pi = S_w \pi_{\varpi_i}$ for some $w \in W$. Then it follows from Lemma 3.5.3 that $\mathrm{cl}(\pi) = \mathrm{cl}(S_w \pi_{\varpi_i}) = S_w(\mathrm{cl}(\pi_{\varpi_i}))$.

We will prove the claim by induction on m. Assume first that $m = 1$. As mentioned above, we have $b = S_w(\mathrm{cl}(\pi_{\varpi_i}))$ for some $w \in W$. This implies that $\mathrm{wt}(b) = w(\mathrm{cl}(\varpi_i))$ by Theorem 3.5.1. But, by assumption, $\mathrm{wt}(b)$ and $\mathrm{cl}(\varpi_i)$ are both contained in the dominant Weyl chamber $\mathring{C}$. Hence it follows that $\mathrm{wt}(b) = \mathrm{cl}(\varpi_i) \in P_{\mathrm{cl}}$. Because $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is, as a $U'_q(\mathfrak{g})$-crystal, isomorphic to the crystal base of a level-zero fundamental $U'_q(\mathfrak{g})$-module (see Propositions 3.4.1 and 3.4.2), we deduce from [Kas3, Theorem 5.17 (iv)] that $b = \mathrm{cl}(\pi_{\varpi_i})$. This proves the claim for the case $m = 1$.

Next we assume that $m > 1$. Since $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ and $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes(m-1)}$ are regular crystals, it follows from [AK, Lemma 1.6(2)] that b_1 and $b_2 \otimes \cdots \otimes b_m$ are also extremal. In addition, we deduce from the proof of [AK, Lemma 1.6(2)] that both of the weights of b_1 and $b_2 \otimes \cdots \otimes b_m$ are elements of the dominant Weyl chamber $\mathring{C}$. Therefore, by the inductive assumption, we get $b_1 = \mathrm{cl}(\pi_{\varpi_i})$ and $b_2 \otimes \cdots \otimes b_m = \mathrm{cl}(\pi_{\varpi_i})^{\otimes(m-1)}$, and hence $b_1 \otimes b_2 \otimes \cdots \otimes b_m = \mathrm{cl}(\pi_{\varpi_i})^{\otimes m}$. This completes the proof of the claim. □

Now, we return to the proof of part (3). Let $\pi \in \mathbb{B}(m\varpi_i)$ be an extremal element. By Lemma 3.5.3, $\mathrm{cl}(\pi) \in \mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ is also extremal. Note that there exists $w \in W$ such that $\mathrm{wt}(S_w(\mathrm{cl}(\pi))) = w(\mathrm{wt}(\mathrm{cl}(\pi))) \in P_{\mathrm{cl}}$ is contained in the dominant Weyl chamber $\mathring{C}$. Let $b_1 \otimes b_2 \otimes \cdots \otimes b_m$ be the element of $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ corresponding to the $S_w(\mathrm{cl}(\pi)) \in \mathbb{B}(m\varpi_i)_{\mathrm{cl}}$ under the isomorphism $\mathbb{B}(m\varpi_i)_{\mathrm{cl}} \cong \mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ (see Lemma 3.4.3 and Proposition 3.4.4). Then, clearly $b_1 \otimes b_2 \otimes \cdots \otimes b_m$ is an extremal element in $\mathbb{B}(\varpi_i)_{\mathrm{cl}}^{\otimes m}$ whose weight is contained in $\mathring{C}$. Therefore, by the claim above, we obtain that $b_1 \otimes b_2 \otimes \cdots \otimes b_m = \mathrm{cl}(\pi_{\varpi_i})^{\otimes m}$, and hence $S_w(\mathrm{cl}(\pi)) = \mathrm{cl}(\pi_{\varpi_i}^{*m})$. Hence it follows from Lemma 3.5.3 that

$$\mathrm{cl}(S_w \pi) = S_w(\mathrm{cl}(\pi)) = \mathrm{cl}(\pi_{\varpi_i}^{*m}).$$

We note that by Proposition 3.2.2, the element $S_w \pi \in \mathbb{B}(m\varpi_i)$ can be written in the form: $S_w \pi = \pi_1 * \pi_2 * \cdots * \pi_m$, where $\pi_l \in \mathbb{B}(\varpi_i)$ for $1 \leqslant l \leqslant m$. Thus we obtain that

$$\begin{aligned} \mathrm{cl}(S_w \pi) &= \mathrm{cl}(\pi_1) * \mathrm{cl}(\pi_2) * \cdots * \mathrm{cl}(\pi_m) \\ &= \mathrm{cl}(\pi_{\varpi_i}^{*m}) = \mathrm{cl}(\pi_{\varpi_i}) * \mathrm{cl}(\pi_{\varpi_i}) * \cdots * \mathrm{cl}(\pi_{\varpi_i}) \quad (m\text{-times}) \end{aligned}$$

and hence $\mathrm{cl}(\pi_l) = \mathrm{cl}(\pi_{\varpi_i})$ for all $1 \leqslant l \leqslant m$. On the other hand, it follows from Lemma 2.1.5 and the definition of LS paths (see Definition 1.4.4) that if $\eta \in \mathbb{B}(\varpi_i)$ satisfies $\mathrm{cl}(\eta) = \mathrm{cl}(\pi_{\varpi_i})$, then $\eta = \pi_{\varpi_i - k d_i \delta}$ for some $k \in \mathbb{Z}$. Applying this to the equality $\mathrm{cl}(\pi_l) = \mathrm{cl}(\pi_{\varpi_i})$ for each $1 \leqslant l \leqslant m$, we see that $\pi_l = \pi_{\varpi_i - k_l d_i \delta}$ for some $k_l \in \mathbb{Z}$, and hence that

$$S_w \pi = \pi_{\varpi_i - k_1 d_i \delta} * \pi_{\varpi_i - k_2 d_i \delta} * \cdots * \pi_{\varpi_i - k_m d_i \delta},$$

where we have $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$ by Lemma 3.2.6. If $w' \in W$ is chosen in such a way that $\varpi_i + k_m d_i \delta = w'(\varpi_i)$, then it follows (see the proof of part (1)) that

$$S_{w'w} \pi = \pi_{\varpi_i - (k_1 - k_m) d_i \delta} * \pi_{\varpi_i - (k_2 - k_m) d_i \delta} * \cdots * \pi_{\varpi_i - (k_{m-1} - k_m) d_i \delta} * \pi_{\varpi_i}.$$

Thus, we have proved part (3). $\square$

The following proposition immediately follows from Lemma 3.5.4.

Proposition 3.5.5. *Each connected component of* $\mathbb{B}(m\varpi_i)$ *is equal to* $\mathbb{B}_\sigma(m\varpi_i)$ *for some* $\sigma \in \mathrm{Par}_{<m}$.

3.6. *Connected components of* $\mathbb{B}(m\varpi_i)$

Lemma 3.6.1. *Let* $k_1, k_2, \ldots, k_{m-1} \in \mathbb{Z}$ *be such that* $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{m-1} \geqslant 0$, *and let* $l_1, l_2, \ldots, l_{m-1} \in \mathbb{Z}$ *be such that* $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_{m-1} \geqslant 0$. *If there exists* $1 \leqslant p \leqslant m-1$ *such that* $k_p \neq l_p$, *then* $\pi_{\varpi_i - k_1 d_i \delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}$ *is not connected to* $\pi_{\varpi_i - l_1 d_i \delta} * \cdots * \pi_{\varpi_i - l_{m-1} d_i \delta} * \pi_{\varpi_i}$ *in the crystal graph of* $\mathbb{B}(m\varpi_i)$.

Proof. Take the maximal $1 \leqslant p \leqslant m-1$ such that $k_p \neq l_p$. We may assume that $k_p > l_p$. Suppose that the assertion is false, i.e., that there exist $j_1, j_2, \ldots, j_n \in I$ such that

$$x_{j_1} x_{j_2} \cdots x_{j_n} (\pi_{\varpi_i - k_1 d_i \delta} * \cdots * \pi_{\varpi_i - k_{m-1} d_i \delta} * \pi_{\varpi_i}) = \pi_{\varpi_i - l_1 d_i \delta} * \cdots * \pi_{\varpi_i - l_{m-1} d_i \delta} * \pi_{\varpi_i},$$

where x_j is either e_j or f_j. Since the concatenation $\pi_{\varpi_i}^{*m} := \pi_{\varpi_i} * \pi_{\varpi_i} * \cdots * \pi_{\varpi_i}$ (m-times) is equal to $\pi_{m\varpi_i}$, and hence is contained in $\mathbb{B}(m\varpi_i)$, we see that $x_{j_1} x_{j_2} \cdots x_{j_n}(\pi_{\varpi_i}^{*m})$ is also contained in $\mathbb{B}(m\varpi_i)$. But, using Lemma 3.3.1, we deduce that

$$
\begin{aligned}
&x_{j_1}x_{j_2}\cdots x_{j_n}(\pi_{\varpi_i}^{*m}) \\
&\quad = x_{j_1}x_{j_2}\cdots x_{j_n}((\pi_{\varpi_i-k_1d_i\delta} * \cdots * \pi_{\varpi_i-k_{m-1}d_i\delta} * \pi_{\varpi_i}) + (\pi_{k_1d_i\delta} * \cdots * \pi_{k_{m-1}d_i\delta} * \pi_0)) \\
&\quad = x_{j_1}x_{j_2}\cdots x_{j_n}(\pi_{\varpi_i-k_1d_i\delta} * \cdots * \pi_{\varpi_i-k_{m-1}d_i\delta} * \pi_{\varpi_i}) + \pi_{k_1d_i\delta} * \cdots * \pi_{k_{m-1}d_i\delta} * \pi_0 \\
&\quad = \pi_{\varpi_i-(l_1-k_1)d_i\delta} * \cdots * \pi_{\varpi_i-(l_{p-1}-k_{p-1})d_i\delta} * \pi_{\varpi_i-(l_p-k_p)d_i\delta} * \pi_{\varpi_i}^{*(m-p)},
\end{aligned}
$$

where we have defined a path $\pi_1 + \pi_2 \in \mathbb{P}$ by: $(\pi_1 + \pi_2)(t) := \pi_1(t) + \pi_2(t)$, $t \in [0, 1]$, for two paths $\pi_1, \pi_2 \in \mathbb{P}$. Since $k_p > l_p$ and $m - p \geqslant 1$, we see from Lemma 3.2.6 that the extreme right-hand side is not contained in $\mathbb{B}(m\varpi_i)$, which is a contradiction. This proves the lemma. $\square$

From the lemma above, we obtain the following.

Proposition 3.6.2. *Let $\sigma, \tau \in \mathrm{Par}_{<m}$. If $\sigma \neq \tau$, then we have $\mathbb{B}_\sigma(m\varpi_i) \neq \mathbb{B}_\tau(m\varpi_i)$.*

3.7. Proof of the main result

Proof of Theorem 3.1.1. From Propositions 3.2.7, 3.5.5, and 3.6.2, we conclude that

$$\mathbb{B}(m\varpi_i) = \bigsqcup_{\sigma \in \mathrm{Par}_{<m}} \mathbb{B}_\sigma(m\varpi_i).$$

Here, using Lemma 3.3.1, we deduce that for each $\sigma \in \mathrm{Par}_{<m}$, the crystal $\mathbb{B}_\sigma(m\varpi_i)$ is isomorphic to the crystal $\mathbb{B}_0(m\varpi_i)$, up to a shift of weight. Namely, we have a bijection $\Phi_\sigma : \mathbb{B}_\sigma(m\varpi_i) \to \mathbb{B}_0(m\varpi_i)$ satisfying the following:

$$\Phi_\sigma(\pi_{\varpi_i-k_1d_i\delta} * \cdots * \pi_{\varpi_i-k_{m-1}d_i\delta} * \pi_{\varpi_i}) = \pi_{\varpi_i}^{*m},$$

$$\Phi_\sigma(e_j\pi) = e_j\Phi_\sigma(\pi), \quad \Phi_\sigma(f_j\pi) = f_j\Phi_\sigma(\pi) \quad \text{for all } \pi \in \mathbb{B}_\sigma(m\varpi_i) \text{ and } j \in I,$$

$$\mathrm{wt}(\Phi_\sigma(\pi)) = \mathrm{wt}(\pi) + |\sigma| d_i\delta \quad \text{for all } \pi \in \mathbb{B}_\sigma(m\varpi_i).$$

Now, define a map $\Psi_\sigma : \mathbb{B}_\sigma(m\varpi_i) \to \mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i)$ for $\sigma \in \mathrm{Par}_{<m}$ by:

$$\Psi_\sigma(\pi) = \sigma \otimes \Phi_\sigma(\pi) \quad \text{for } \pi \in \mathbb{B}_\sigma(m\varpi_i)$$

and then define a map $\Psi : \mathbb{B}(m\varpi_i) \to \mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i)$ by: $\Psi(\pi) = \Psi_\sigma(\pi)$ if $\pi \in \mathbb{B}_\sigma(m\varpi_i)$ for $\sigma \in \mathrm{Par}_{<m}$. We can easily check that Ψ is an isomorphism between the crystals $\mathbb{B}(m\varpi_i)$ and $\mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i)$, thereby completing the proof of Theorem 3.1.1. $\square$

3.8. Relation to the crystal base $\mathcal{B}(m\varpi_i)$

Here we use the notation of §2.2. For $m \geqslant 1$, we denote by $\mathcal{B}(m\varpi_i)$ the crystal base of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$. Let $\mathcal{B}_0(m\varpi_i)$ be the connected component of the crystal base $\mathcal{B}(m\varpi_i)$ containing the canonical extremal element $u_{m\varpi_i}$ (see [Kas3, §3.1]). We know from [NS, Corollary 5.3] that the connected component $\mathcal{B}_0(m\varpi_i)$ is, as a crystal, isomorphic to the connected component $\mathbb{B}_0(m\varpi_i)$ (see Remarks 1.2.1 and 1.4.7). Namely, $\mathcal{B}_0(m\varpi_i) \cong \mathbb{B}_0(m\varpi_i)$ as crystals.

On the other hand, we know from [BN, Theorem 4.16 (i)] that the crystal base $\mathcal{B}(m\varpi_i)$ is, as a crystal, isomorphic to the tensor product $\mathrm{Par}_{<m} \otimes \mathcal{B}_0(m\varpi_i)$. Therefore, by combining these facts with Theorem 3.1.1, we obtain the following corollary.

Corollary 3.8.1. *The crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ of extremal weight $m\varpi_i$.*

3.9. Comments

In [GL], they studied the loop module $\widehat{V}(\varpi_{i;m})$ of fundamental type over $U_q(\mathfrak{g})$ in the case where $\mathfrak{g}$ is an untwisted affine Lie algebra. Let $\widehat{V}(\varpi_{i;m})^{(k)}$, $0 \leqslant k \leqslant m-1$, be the irreducible components of the loop module $\widehat{V}(\varpi_{i;m})$ (see [GL, §4.2]). They showed that each $\widehat{V}(\varpi_{i;m})^{(k)}$ admits a z-crystal base $\widehat{\mathcal{B}}(\varpi_{i;m})^{(k)}$ for $0 \leqslant k \leqslant m-1$, where z is an mth primitive root of unity. Furthermore, they proved that for $0 \leqslant k \leqslant m-1$, the associated crystal of $\widehat{\mathcal{B}}(\varpi_{i;m})^{(k)}$ is, as a crystal, isomorphic to the connected component of $\mathbb{P}/\sim$ containing the straight line $\pi_{m\varpi_i+k\delta}$. Here we note that this connected component is isomorphic to the connected component $\mathbb{B}_0(m\varpi_i + k\delta)$ of $\mathbb{B}(m\varpi_i + k\delta)$ (which is regarded as a subset of $\mathbb{P}$) containing the straight line $\pi_{m\varpi_i+k\delta}$ (see Remark 1.4.7).

If $m = 1$, then $\widehat{\mathcal{B}}(\varpi_{i;1})^{(0)}$ is an ordinary crystal, which is isomorphic to $\mathbb{B}(\varpi_i)$ as a crystal. In fact, we know from [Kas3, Theorem 5.17 (viii)] that the loop module $\widehat{V}(\varpi_{i;1}) = \widehat{V}(\varpi_{i;1})^{(0)}$ is isomorphic to the extremal weight module $V(\varpi_i)$ (see also the Introduction of [GL]).

On the other hand, if $m \geqslant 2$, then we can deduce that for $k = 0, 1, \ldots, m-1$:

(1) if k is divisible by d_i, then the crystal $\mathbb{B}(m\varpi_i)$ has infinitely many connected components isomorphic to $\mathbb{B}_0(m\varpi_i + k\delta)$ as a crystal. Indeed, it is easy to see that $\mathbb{B}_\sigma(m\varpi_i) \cong \mathbb{B}_0(m\varpi_i + k\delta)$ as crystals if $-|\sigma| \equiv k/d_i \bmod m$ (however, any connected component of $\mathbb{B}(m\varpi_i)$ does not agree with $\mathbb{B}_0(m\varpi_i + k\delta)$ as a set of paths unless $k = 0$);
(2) if k is not divisible by d_i, then any connected component of $\mathbb{B}(m\varpi_i)$ is not isomorphic to $\mathbb{B}_0(m\varpi_i + k\delta)$ as a crystal.

Hence, it is likely that the loop module $\widehat{V}(\varpi_{i;m})$ does not coincide with the extremal weight module $V(m\varpi_i)$ for $m \geqslant 2$ (but, they are closely related to each other).

Acknowledgments

We would like to express our sincere thanks to Professor Masato Okado for fruitful discussions.

References

[AK] T. Akasaka, M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. 33 (1997) 839–867.

[B] J. Beck, Crystal structure of level zero extremal weight modules, Lett. Math. Phys. 61 (2002) 221–229.

[BN] J. Beck, H. Nakajima, Crystal bases and two-sided cells of quantum affine algebras, Duke Math. J. 123 (2004) 335–402.

[G] J. Greenstein, Littelmann's path crystal and combinatorics of certain integrable $\widehat{sl_{\ell+1}}$ modules of level zero, J. Algebra 269 (2003) 347–372.

[GL] J. Greenstein, P. Lamprou, Path model for quantum loop modules of fundamental type, Internat. Math. Res. Not. 2004(14) 675–711.

[Kac] V.G. Kac, Infinite Dimensional Lie Algebras, 3rd ed., Cambridge University Press, Cambridge, UK, 1990.

[Kas1] M. Kashiwara, Crystal bases of modified quantized enveloping algebra, Duke Math. J. 73 (1994) 383–413.

[Kas2] M. Kashiwara, On crystal bases, in: B.N. Allison, G.H. Cliff (Eds.), Representations of Groups, CMS Conf. Proc. vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.

[Kas3] M. Kashiwara, On level-zero representations of quantized affine algebras, Duke Math. J. 112 (2002) 117–175.

[L1] P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1994) 329–346.

[L2] P. Littelmann, Paths and root operators in representation theory, Ann. of Math. (2) 142 (1995) 499–525.

[NS] S. Naito, D. Sagaki, Path model for a level-zero extremal weight module over a quantum affine algebra, Internat. Math. Res. Not. 2003(32) 1731–1754.

[N] H. Nakajima, Extremal weight modules of quantum affine algebras, in: T. Shoji, M. Kashiwara, et al. (Eds.), Representation Theory of Algebraic Groups and Quantum Groups, Advanced Studies in Pure Mathematics, vol. 40, Mathematical Society of Japan, 2004, pp. 343–369.